MODELS FOR THE FIELD ARTILLERY DESTRUCTION MISSION

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THESIS

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DESTRUCTION MISSION

by

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ABSTRACT

The purpose of this thesis is to mathematically model the Field Artillery Destruction Mission. The author felt that advances in technology might allow the development of procedures that are more efficient than those currently in use. In particular TACFIRE, a computer based fire direction center, and the laser range-finder were taken into consideration. Using the capabilities resulting from these technological advances, a classical and Bayesian model of the destruction mission was developed. Each model was analyzed and conclusions were drawn regarding the appropriate model to use in a given situation.

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I. INTRODUCTION

The purpose of this thesis is to mathematically model the Field Artillery Destruction Mission with a view toward developing a best procedure for conducting such a mission. A best procedure is considered to be one which destroys a given target using, on the average, the least number of rounds.

Current doctrine, described in Reference 1, calls for one weapon to fire a sequence of rounds at the target. Using visual sensings of round impact points by a forward observer, the fire direction center computes elevation and direction data for the weapon. These data, when set on the weapon, determine an aim point around which the round will impact. Where the round will actually impact is a function of its ballistic distribution pattern which is random and is further discussed in Chapter II. The mission is conducted in two phases. Phase one is controlled by the forward observer and is completed when the observer has split a 100 meter range bracket of the target. A 100 meter range bracket is established when the observer makes a 100 meter shift that results in an impact sensing opposite from the round on which the shift was based. For example, suppose the third round is sensed as short of the target. If, after increasing the range by 100 meters, the fourth round impacts over the target, a 100 meter range bracket has been obtained. Phase two is controlled by the fire direction

center. Using a computation procedure based on forward observer sensings and the round by round distribution pattern, the fire direction center computes a sequence of elevation and direction settings to be placed on the weapon. This phase continues until the target is hit. Important aspects of the current procedure are:

1. Observer sensings are given to the fire direction center as over, short, right, or left of the target. The observer does not attempt to estimate how far from the target each round impacts.

2. Elevation and direction settings are computed manually by means of firing charts and specially designed slide rules.

3. At each step in the process, all previous information regarding impact points is not taken into consideration when determining new elevation and direction settings.

Recent technological developments may allow the development of procedures that are more efficient than those currently in use. In particular, TACFIRE, which is a computer based fire direction center, will provide a faster and more accurate determination of elevation and direction settings than the manual fire direction center. Laser rangefinders will enable the observer to provide the fire direction center with information relative to how far from the target each round impacts. Both TACFIRE and the laser rangefinder are being phased into artillery units at the present time.

The models developed in this paper take into consideration the increased capabilities resulting from TACFIRE and the laser range-finder. In Chapter II a general model of the destruction mission is developed. In Chapters III and IV, two specific models are considered. A comparison of these models is given in Chapter V, and several conclusions and recommendations are presented in Chapter VI.

II. GENERAL MODEL OF THE DESTRUCTION MISSION

A. GENERAL

In this Chapter a general model of the destruction mission will be developed. The US Army 8 inch howitzer is assumed to be the weapon system used for the mission so that numerical examples can be presented for clarity or to emphasize key points. The model is general, however, and applies to any howitzer currently in the artillery inventory. Two ballistic errors are involved when an attempt is made to hit a target with an artillery round: range and deflection errors. In comparison with range errors, deflection errors are nearly always insignificant. For example, at a gun to target range of 12,000 meters the probable range error is 21 meters whereas the probable deflection error is only three meters [Reference 2]. Targets appropriate for a destruction mission are stationary and typical dimensions are: bridge, 10 x 200 meters; pillbox, 10 x 10 meters; fortification, 20 x 20 meters. Since the ballistic distribution of artillery rounds is normal, 96 percent of the rounds fired will impact within three probable errors of the target when the aim point is centered on the target [Reference 1]. Ballistic range error, therefore, is considered most critical, and in the development of the model ballistic deflection error will be ignored. The basic considerations given below, however, could be applied to deflection errors if desired.

B. ASSUMPTIONS

1. A target is destroyed when a round lands directly on the target. By eliminating the possibility of target destruction by a near miss, the mathematical development of the model can be simplified.

2. The round impact points fall in a normal distribution pattern around the aim point [Reference 1].

3. When the observer asks for a shift in the round impact point, he gets exactly what he asks for. For example, if the observer asks for a reduction of 100 meters in range, the fire direction center will develop new range and direction data that, when placed on the howitzer, will cause the mean of the normal distribution to move 100 meters in the appropriate direction. This assumption ignores possible fire direction center and gun crew errors. With the introduction of TACFIRE, fire direction errors should be minimal. Gun crew errors, while they do exist, are rare and would be inherent to any procedure. This error is a function of crew training and command supervision and it was not considered feasible to attempt to incorporate this error into the model.

4. The observer is capable of providing the fire direction center the impact point of each round, relative to the target, without error. The extent of observer error will be a function of the observers capability to properly use the laser range-finder and the inherent error in the range-finder itself. To date, data relative to the size

and distribution of these errors has not been developed. Because it is anticipated that such errors will be small, and for the sake of mathematical tractibility, these errors will be ignored. Sensitivity of this model to observer error is an area that should be pursued in future research.

5. The initial impact of the first round will be within 400 meters of the target. Based on the authors' experience, most observers are currently capable of locating the target with sufficient accuracy so that the initial round will land within 400 meters of the target.

6. When shifts are made the mean of the normal distribution changes but the range probable error does not. For the 8 inch howitzer the maximum change in range probable error per 1000 meter change in range is 1.5 meters [Reference 2]. Since the range changes under consideration should virtually never exceed 400 meters, this assumption is considered to be reasonable.

C. MODEL

The destruction mission can be viewed as a series of stages and can be graphically presented as shown in Figure 1

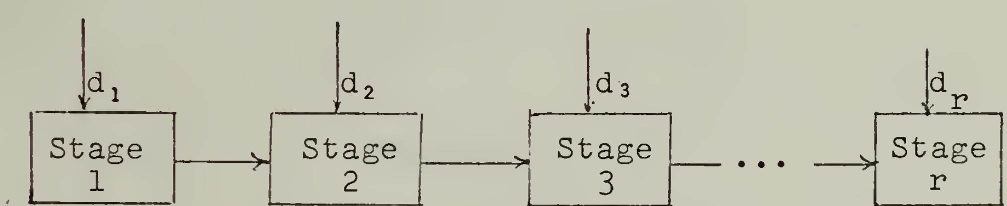

Figure 1.

In each stage i, one round is fired at the target, and the decision-maker makes a decision, d_i. The term decision-maker will apply to the person or thing that makes the decision. For example, the decision may be made by the fire direction officer or in some cases by the computer, based on a set of decision rules. The decision made at any given stage can result in one of the following actions:

1. If the round hit the target, the mission has been successful and is terminated.

2. If the round did not hit the target the decision-maker can fire another round at the same gun setting or take an action that will move the mean of the ballistic distribution. When another round is fired a transition is made to the next stage in the model.

The model also makes use of a shifting coordinate system. This concept provides an orderly method of mathematically depicting information resulting from previously fired rounds. For example, consider stage one. Suppose the first round impacts as shown in Figure 2.

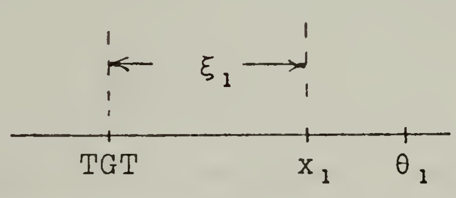

Figure 2.

The symbol x_1 represents the impact point of round number one in the first stage and θ_1 represents the mean of the ballistic distribution of round one. Suppose further that

the decision-maker makes the decision at stage one to decrease the range by a distance ξ_1. Round two is then fired and the mean of its ballistic distribution is θ_2. The situation at stage two can now be presented as shown in Figure 3.

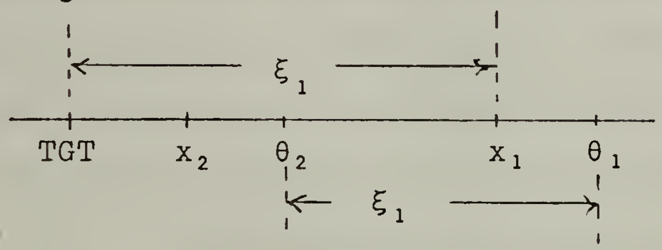

Figure 3.

Assumptions three and six allow the impact point, x_1, to be easily transformed so that it can be expressed in terms of the ballistic distribution of x_2. This situation is depicted in Figure 4.

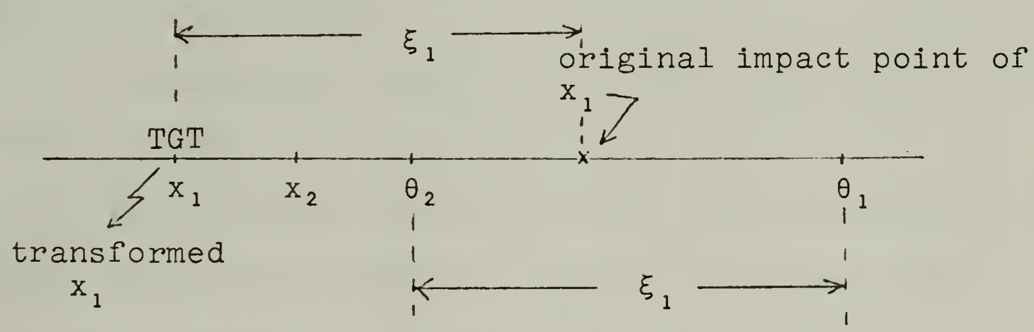

Figure 4.

As a result of this transformation, the decision-maker can be mathematically presented with a random sample of two elements from a ballistic distribution whose mean is θ_2, upon which to base a decision. In a similar manner round two can be transformed to the ballistic distribution of

round one. In this model, therefore, each stage is identified with a specific ballistic distribution and coordinate system. It is not required, however, that each distribution be unique. For example, if a zero shift is made, so that the next round is fired at the same gun setting, the ballistic distribution of this round will not change. To mathematically depict the changing coordinate system concept the following notation is established.

X_i^r - A random variable denoting the impact point of the i^{th} round in the r^{th} coordinate system. For r=1, the X_i^1's are assumed to be independent, identically distributed random variables and are distributed $N(\theta,\sigma^2)$. The mean of the distribution is denoted by θ, and σ^2 denotes the variance of the distribution. The variance is determined from the firing tables for a particular weapon and is a function of the gun target range.

x_i^r - Actual impact point of the i^{th} round in the r^{th} coordinate system.

ξ_i - The distance the mean of the ballistic distribution will be moved at stage i. ξ_i can be equal to zero. Using this notation the situation depicted in Figure 4 can be represented as shown in Figure 5.

Stage 2

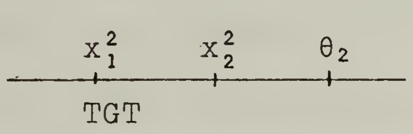

Figure 5.

D. OBJECTIVE

The objective of the destruction mission is to destroy the target with the expenditure of the least number of rounds. The purpose of this section is to determine a set of decision rules that will be consistent with this objective. Since the number of rounds required to hit the target for the first time is a random variable, a rule that will minimize the expected value of this random variable is chosen as a criterion for choice among decision rules.

In the following development, let:

$s = (s_1(x_1), s_2(x_1,x_2),\ldots)$ denote a sequential mission strategy, where $s_j(x_1,\ldots,x_j)$ is the shift made prior to firing the $(j+1)$st round and S will denote the class of all such strategies.

$P(s) = \{P_1(s), P_2(s),\ldots\}$ is a vector of conditional hit probabilities using s. $P_i(s)$ is the conditional probability of hitting the target in the i^{th} stage using strategy s, given that the target was not hit in the previous $(i-1)$ stages.

$F = \{P(s);\ s\varepsilon S\}$ is a set of all feasible conditional hit probability vectors.

J denotes a random variable representing the number of rounds required to hit the target for the first time.

Q denotes the set of all distribution functions of the random variable J and $q(s)\varepsilon Q$.

THEOREM 1. For some strategy $s^*\varepsilon S$, the expected number of rounds required to hit the target for the first time,

$E(J)$, is minimized if and only if the probability of hitting the target in stage i, given that the target has not been hit in a previous stage, $P_i(s^*)$, is equal to the maximum over all $s \varepsilon S$ of $P_i(s)$. This can be mathematically expressed as follows:

(1)

$$\left\{E_{q(s^*)}(J) = \min_{Q} E(J)\right\} \Longleftrightarrow \left\{P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s) \cdot \forall i \varepsilon \{1,2,\ldots\}\right\} .$$

Proof: The proof will be given in three parts.

1. The event $\{J>1\}$ occurs if and only if the target is not hit on the first round. Similarly the event $\{J>n\}$ occurs if and only if the target is not hit before $(n+1)$ rounds have been fired, given that the target was not hit on any of the previous n rounds.

For the case where $n=3$, the $P(J>3)$ can be expressed as follows:

Let A denote the event that the target is missed on the first round, B the event that the target is missed on the second round, and C the event that the target is missed on the third round.

$$\begin{aligned} \text{Then}\quad P(J>3) &= P(A)P(B|A)P(C|A\cap B) \\ &= P(A)\ \frac{P(A\cap B)}{P(A)}\ \frac{P(A\cap B\cap C)}{P(A\cap B)} \\ &= P(A\cap B\cap C). \end{aligned}$$

In terms of $P_i(s)$

$$P(J>3) = (1 - P_1(s))(1 - P_2(s))(1 - P_3(s)).$$

This expression can be generalized to any value of n so that,

(2)

$$P(J>n) = \prod_{i=1}^{n} (1 - P_i(s)).$$

2. Show:

$$\left\{P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,2,...\}\right\}$$

$$\Longleftrightarrow \left\{P_{q(s^*)}[J>n] = \min_{Q} P[J>n]\right\}.$$

First by induction it is shown that:

$$\left\{P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,2,...\}\right\}$$

$$\Longrightarrow \left\{P_{q(s^*)}[J>n] = \min_{Q} P[J>n]\right\}.$$

a. For n=1 $(1 - P_1(s)) = P[J>1]$ so that if $s^*\varepsilon S$ gives

$$P_1(s^*) = \max_{P(s)\varepsilon F} P_1(s),$$

then $q(s^*)$ will minimize

$$P[J>1] \text{ over } Q.$$

b. Assume the above relationship holds

$$\forall i\varepsilon\{1,2,...,n-1\}.$$

c. Show true $\forall i\varepsilon\{1,2,...,n\}$. From equation (2),

$$(1 - P_n(s)) \prod_{i=1}^{n-1} (1 - P_i(s)) = (1 - P_n(s))P[J>n-1]$$

$$= P[J>n] .$$

By assumption

$$\left\{ P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,\ldots,n-1\} \right\}$$

$$\Longrightarrow \left\{ P_{q(s^*)}[J>n-1] = \min_{Q} P[J>n-1] \right\},$$

where

$$P[J>n-1] \geq 0 \text{ and } 0 \leq P_n(s) \leq 1.$$

This implies that if s* gives

$$P_n(s^*) = \max_{P(s)\varepsilon F} P_n(s),$$

then q(s*) will minimize P[J>n] over Q, that is,

$$\left\{ P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\{1,\ldots,n\} \right\}$$

$$\Longrightarrow \left\{ P_{q(s^*)}[J>n] = \min_{Q} P[J>n] \right\}.$$

Next by induction it is shown that,

$$\left\{ P_{q(s^*)}[J>n] = \min_{Q} P[J>n] \right\}$$

$$\Longrightarrow \left\{ P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,\ldots,n\} \right\}$$

a. If $P(J>1) = (1 - P_1(s))$ then $P(J>1)$ will be minimized over Q provided that $P_1(s^*) = \max_{P(s)\varepsilon F} P_1(s)$.

b. Assume the above relationship holds $\forall i\varepsilon\{1,\ldots,n-1\}$.

c. Show true $\forall i\varepsilon\{1,\ldots,n\}$. From equation (2),

$$P[J>n] = (1 - P_n(s))\ P[J>n-1].$$

By assumption

$$\left\{P_{q(s^*)}[J>n-1] = \min_{Q} P[J>n-1]\right\}$$

$$\Rightarrow \left\{P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,\ldots,n-1\}\right\}$$

where

$$P(J>n-1) \geq 0 \text{ and } 0 \leq P_n(s) \leq 1.$$

This implies that $P(J>n)$ will be minimized over Q provided that $P_n(s^*) = \max_{P(s)\varepsilon F} P_n(s)$. Therefore,

$$(3)\quad \left\{P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,\ldots,n\}\right\}$$

$$\Leftrightarrow \left\{P_{q(s^*)}[J>n] = \min_{Q} P[J>n]\right\}.$$

3. Since J is a non-negative integer valued random variable the expected value of J can be written as

$$E(J) = \sum_{n=1}^{\infty} P(J>n) \text{ or}$$

$$\min_{Q} E(J) = \min_{Q} \sum_{n=1}^{\infty} P(J>n).$$

If $P(J>n)$ can be independently minimized for each n, the above equation can be written in the form,

$$(4)\quad \min_{Q} E(J) = \sum_{n=1}^{\infty} \min_{Q} P[J>n].$$

By equation (3), $P(J>n)$ is minimized over Q when,

$$P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,\ldots,n\}.$$

Therefore,

$$\left\{E_{q(s^*)}(J) = \min_{Q} E(J)\right\}$$

$$\Longleftrightarrow \left\{P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,\ldots,n\}\right\} .$$

This completes the proof of the Theorem.

This means that in order to minimize the number of rounds required to hit the target for the first time, at each decision point, the decision-maker should evaluate the information resulting from all previously fired rounds and take the action that will maximize the conditional probability of hit on the next round.

For ease of notation let P_i denote the conditional probability of hit on the i^{th} round, given that the target was not hit on the previous (i-1) rounds. It will be understood that a mission strategy s* that results in

$$P_i(s^*) = \max_{P(s)\varepsilon F} P_i(s),\ \forall i\varepsilon\{1,2,\ldots\}$$

is being used.

THEOREM 2. When a round from a normal ballistic distribution with fixed variance is fired at the center of a target, the probability of hit will be a maximum when the mean of the distribution is the target center.

Proof: Let Z be a random variable representing the impact point of the round to be fired. The probability of hit will be

$$P(\text{hit}) = \int_{-a}^{a} \frac{1}{\sqrt{2\pi}\,\sigma} \exp - \frac{1}{2} \frac{(Z-\theta)^2}{\sigma^2}\ dZ$$

where 2a is a measure of the size of the target which is centered at zero. To find the value of θ that maximizes the probability of hit, the derivative with respect to θ of the above expression is set equal to zero, and the resulting equation is solved for θ. Interchanging order of integration and differentiation results in

$$\int_{-a}^{a} \frac{d}{d\theta} \left\{ \frac{1}{\sqrt{2\pi}\ \sigma} \exp - \frac{1}{2} \left(\frac{Z-\theta}{\sigma} \right)^2 dZ \right\}$$

$$= -\int_{-a}^{a} \frac{1}{\sqrt{2\pi}\ \sigma} \exp - \frac{1}{2} \left(\frac{Z-\theta}{\sigma} \right)^2 - \frac{(Z-\theta)}{\sigma^2} dZ$$

$$= - \frac{1}{\sqrt{2\pi}\ \sigma} \left\{ \exp - \frac{1}{2} \left(\frac{a-\theta}{\sigma^2} \right)^2 - \exp - \frac{1}{2} \left(\frac{-a-\theta}{\sigma} \right)^2 \right\}$$

which is zero if and only if $\theta = 0$, since $a \neq 0$. Q.E.D.

This means that if the ballistic distribution of a round is normal, the decision-maker, in order to maximize the probability of hitting a target centered at zero, should take an action that will result in the mean of the distribution being equal to zero. Thus, from Theorem 1, it follows that an optimal strategy s* is one in which each shift, $s_i^*(x_1,\ldots,x_i)$, will result in the mean of the distribution of the (i+1)st round being equal to zero. For ease of notation let $\xi_i = s_i^*(x_1,\ldots,x_i)$.

III. MODEL I

A. DESCRIPTION OF THE MODEL

In this section a mathematical model of the destruction mission and a decision rule resulting from an analysis of the model will be presented. This model is a particular version of the general model. In addition to the assumptions listed in Chapter II, model I requires the assumption that the first round fired does not hit the target. This assumption is considered reasonable in that if the first round does hit the target, the mission is terminated and there is no need to proceed further. Additionally, the nature of the conditional random variables X_i^r will be discussed. Throughout the following discussion X_i^r will be from a probability distribution $f(X_i^r | x_1^{r-1}, \ldots, x_{i-1}^{r-1})$ and will be referred to as a conditional random variable.

The impact point of the first round, X_1^1, is a random variable which is distributed $N(\theta, \sigma^2)$. Suppose a realization of X_1^1 results in the situation shown in Figure 6.

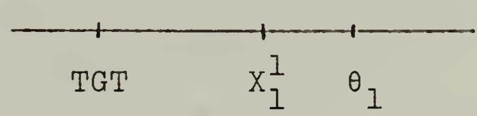

Figure 6.

In Figure 6, X_1^1 represents the observed impact point and θ_1 is the mean of the ballistic distribution of X_1^1. If X_2^2 is fired at the same gun setting as X_1^1, it will also, by assumption, be distributed $N(\theta_1, \sigma^2)$, independent of X_1^1. In

accordance with Theorem 2, to maximize the conditional probability of hit on the second round, the mean of X_2^2 must be equal to zero. Let X_2^2 be distributed $N(\theta_1+\xi_1,\gamma_2^2)$ where ξ_1 is the shift required to make the mean of X_2^2 equal zero and γ_2^2 is the variance of X_2^2. This implies that the value of ξ_1 should be equal to $-\theta_1$. Although the value of θ_1 is unknown, it may be estimated from the sample data.

To illustrate further, consider stage four and a possible graphical representation as shown in Figure 7.

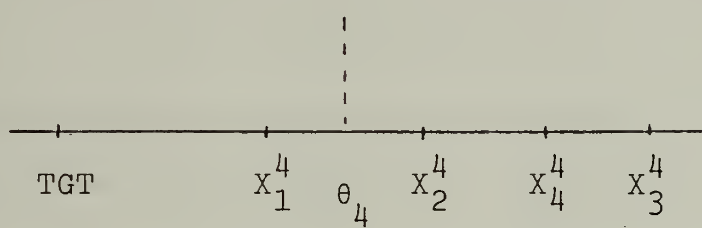

Figure 7.

The next round, X_5^5, will be distributed $N(\theta_4+\xi_4,\gamma_5^2)$. As indicated above, to maximize the probability of hit, the value of ξ_4 should be equal to $-\theta_4$. To estimate θ_4, the decision-maker has four sample points available. Suppose the statistic $Y_4 = \frac{1}{4} \sum_{i=1}^{4} X_i^4$ is used to estimate θ_4. This statistic is unbiased and its variance reaches the Rao-Cramer lower bound [Reference 3]. For this example it seems plausible that the decision-maker should make a shift $\xi_4 = -\frac{1}{4}(x_1^4+...+x_4^4)$. This can be generalized so that $\xi_n = -1/n\ (x_1^n+...+x_n^n)$ and by making use of assumption three, Chapter II, it can be shown that this shift reduces $\xi_n = -1/n\ (x_n^n)$.

Theorem 3. If a shift of $\xi_n = -1/n(x_1^n+\ldots+x_n^n)$ is made at each stage n, ξ_n is only dependent on the most recent round fired, X_n^n, and the total number of rounds fired, n:

(5)

$$\xi_n = -1/n(x_n^n).$$

Proof: Suppose a shift of $\xi_n = -1/n(x_1^n+\ldots+x_n^n)$ is made prior to firing the (n+1)st round and suppose the first round has been fired. Then,

$$\xi_1 = -x_1^1 \text{ or } x_1^2 = 0.$$

Stage two can then be represented as shown in Figure 8, where x_1^1 has been transformed into coordinate system two.

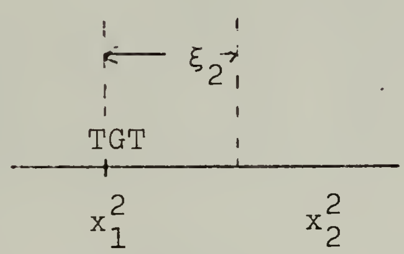

Figure 8.

From Figure 8 $\xi_2 = -\frac{1}{2}(x_1^2 + x_2^2) = -\frac{1}{2}x_2^2$. After round three has been fired, stage three can be represented as shown in Figure 9 where x_1^2 and x_2^2 have been transformed into coordinate system three.

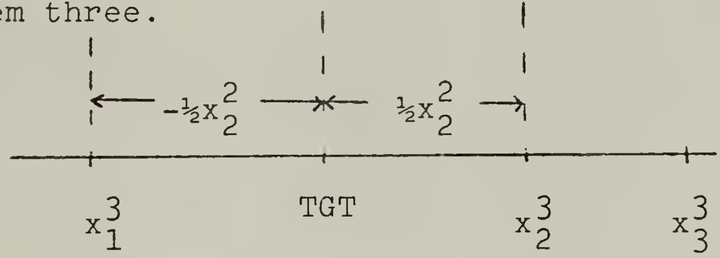

Figure 9.

Thus $\xi_3 = -1/3(x_1^3+x_2^3+x_3^3) = -1/3(-\frac{1}{2}x_2^2+\frac{1}{2}x_2^2+x_3^3) = -1/3(x_3^3)$, and so on for ξ_4, ξ_5, ... QED.

Based on a shift of $\xi_n = -1/n(x_n^n)$, the distribution $f(X_n^n|x_1^n,\ldots,x_{n-1}^{n-1})$ can be determined. As explained in Chapter II it is convenient to transform the sample points to various coordinate systems. In other words, conditional random variables X_i^n, from any coordinate system n, can be expressed in terms of conditional random variables in coordinate system one. The distributions of these conditional random variables can be expressed as shown below.

X_1^1 is distributed $N(\theta_1,\sigma^2)$.

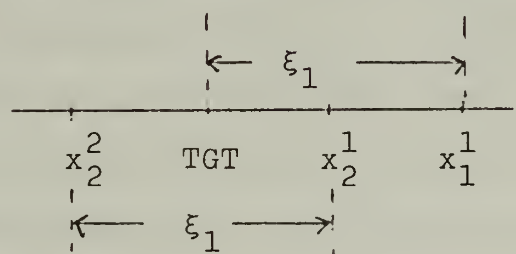

Figure 10.

From Figure 10, X_2^2 can be expressed in terms of where the second round would have impacted if it had been fired in coordinate system one, plus the shift ξ_1. This results in the following expressions.

(6)

$$X_2^2 = X_2^1 + \xi_1 = X_2^1 - X_1^1 \sim N(0,2\sigma^2)$$

$$X_3^3 = X_3^1 + \xi_1 + \xi_2 = X_3^1 - \frac{(X_1^1 + X_2^1)}{2} \sim N(0,\sigma^2+\sigma^2/2)$$

$$X_4^4 = X_4^1 + \xi_1 + \xi_2 + \xi_3 = X_4^1 - \frac{1}{3}(X_1^1+X_2^1+X_3^1) \sim N(0,\sigma^2+\sigma^2/3)$$

.
.
.

$$\vdots$$

$$X_n^n = X_n^1 + \sum_{i=1}^{n-1} \xi_i = X_n^1 - \frac{1}{n-1} \sum_{i=1}^{n-1} X_i^1 \sim N(0,\sigma^2+\sigma^2/n-1).$$

Thus for all n>1 the conditional random variable X_n^n has a normal distribution with a mean of zero and variance $\sigma^2+\sigma^2/n-1$.

The variance of this conditional random variable justifies the use of Y_n as the estimator for θ. The variance can be written in the form $\sigma^2+\sigma^2_{Y_n}$. As stated above, the variance of Y_n reaches the Rao-Cramer lower bound and as a result no other unbiased statistic has a smaller variance. Therefore, by making a shift $\xi_n = -1/n(x_n^n)$ it follows from Theorem 2 that an action has been taken that will maximize the probability of hit on the (n+1)st round, and by the criterion established for this mission the action is in fact optimal.

By means of the following theorems it follows that the conditional random variables, X_n^n, n=2,..., are independent.

Theorem 4. If the p x 1 vector X is distributed normally with mean μ and covariance V and if B is a q x p matrix ($q \leq p$) of rank q, the vector Z = BX is distributed as the q-variate normal with mean $B\mu$ and covariance BVB' [Reference 4].

Theorem 5. If Z has the multivariate normal distribution with mean μ and covariance V, then the components

Z_i are jointly independent if and only if V is diagonal [Reference 4].

Theorem 6. The nonzero rows of a matrix in echelon form are linearly independent [Reference 5].

The following development shows that X_2^2, X_3^3, and X_4^4 are independent random variables. Equations (6) can be written in the form,

$$BX = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \\ X_4^1 \end{pmatrix} = \begin{pmatrix} X^2 \\ X_3^3 \\ X_4^4 \end{pmatrix} = Z$$

By assumption, the vector X has mean

$$\mu = \begin{pmatrix} \theta \\ \theta \\ \theta \\ \theta \end{pmatrix}$$

and variance-covariance matrix

$$V = \begin{pmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{pmatrix}.$$

The matrix B can be reduced to the echelon form

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 3 & -3 \end{pmatrix},$$

and by Theorem 6 is of full rank.

By Theorem 4 the vector Z has a covariance matrix equal to

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\sigma^2 & 0 & 0 \\ 0 & 3\sigma^2/2 & 0 \\ 0 & 0 & 4\sigma^2/3 \end{pmatrix},$$

and by Theorem 5 the components of the vector Z are jointly independent. This development can be expanded to a Z vector equal to $(X_2^2,\ldots,X_n^n)'$ with the conclusion that the conditional random variables, $X_2^2,\ldots,X_n^n$ are jointly independent for $n \geq 2$.

B. RESULTS

1. Decision Rule

From an analysis of model I a decision rule requiring a shift $\xi_n = -1/n(x_n^n)$, at the n^{th} stage, $n = 1,2,\ldots,$ was produced. Using this rule the conditional probability of hit, for each round fired, is maximized. Additionally, by maximizing the conditional probability of hit for each round fired, the overall objective of minimizing the expected number of rounds required to hit the target for the first time is achieved.

2. Distribution of X_n^n

The conditional random variable, X_n^n, representing the impact point of the n^{th} round in the n^{th} coordinate system was shown to have a normal distribution with mean zero

and variance $\sigma^2+\sigma^2/n-1$. Since the conditional random variables, $X_2^2,\ldots,X_n^n$ were shown to be independent for $n\geq 2$, equation (4) of Theorem 1 applies to this model.

IV. MODEL II

A. DESCRIPTION OF THE MODEL

This model has the same basic structure as the general model. It differs from the general model and model I in that the mean of the ballistic distribution, in coordinate system one, is considered to be a random variable, Θ_1. As a result of past experience and knowledge of the firing conditions, the decision-maker may have some idea of where the mean of the first round fired will be in relation to the target. Based on the author's experience it is reasonable to assume that the actual mean of the ballistic distribution is as likely to be over the target as short of it. It is also more likely that the mean will be relatively close to the target, say within ±100 meters, than far from the target. A $N(0,\tau^2)$ distribution seems to provide a reasonable representation of this situation. The variance of the distribution, τ^2, is a measure of how close the actual ballistic mean, θ_1, is to the target. For example, if a registration has been conducted prior to firing a destruction mission and if the observer is using a laser range-finder, τ^2 should be small in relation to its value if a registration had not been conducted and a laser range-finder was not used. The value of τ^2 is, therefore, a subjective measure and is based on past experience or experimental data. It is assumed, therefore, that a $N(0,\tau^2)$ prior probability distribution, with density $g(\theta_1)$, can be assigned to Θ_1.

The following mathematical derivations will establish the posterior distribution of Θ, $f(\Theta|x_x^1,\ldots,x_n^1)$, and the conditional distribution of the n^{th} round given the impact points of the previous (n-1) rounds, $f_\theta(X_n^1|x_1^1,\ldots,x_{n-1}^1)$. The subscript θ is used to emphasize the fact that this distribution is dependent on the prior distribution of Θ. In summary, $\Theta \sim N(\mu,\tau^2)$, $Y_n = 1/n \sum_{i=1}^{n} X_i^1$ is distributed $N(\theta,\sigma^2/n)$, and $h(\Theta,Y_n)$ denotes the joint distribution of Θ and Y_n.

The joint distribution of Θ and Y_n is derived as follows:

$$h(\Theta,Y_n) = f(Y_n|\theta)g(\theta)$$

$$= \left\{\frac{1}{\sqrt{2\pi}\ \sigma/\sqrt{n}} \exp -\frac{1}{2}\ \frac{(Y_n-\theta)^2}{\sigma^2/n}\right\}\left\{\frac{1}{\sqrt{2\pi}\ \tau} \exp -\frac{1}{2}\ \frac{(\theta-\mu)^2}{\tau^2}\right\}$$

$$= \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left[\frac{-(\theta-\mu)^2}{2\tau^2} - \frac{(Y_n-\theta)^2}{2\sigma^2/n}\right] .$$

Letting

$$(Y_n-\theta)^2 = (Y_n-\mu+\mu-\theta)^2 = (\theta-\mu)^2 - 2(\theta-\mu)(Y_n-\mu) + (Y_n-\mu)^2$$

and rearranging terms yields

$$h(\Theta,Y_n) = \frac{\sqrt{n}}{2\pi\sigma\tau} \exp -\frac{1}{2}\left\{\frac{n\tau^2\ \sigma^2}{\sigma^2}\left[\frac{(\theta-\mu)^2}{\tau^2} - \left(\frac{2\tau\sqrt{n}}{\sqrt{n\tau^2+\sigma^2}}\right) \cdot \left(\frac{(\theta-\mu)(Y_n-\mu)\ n}{\tau\sqrt{n\tau^2+\sigma^2}}\right) + \frac{n(Y_n-\mu)^2}{n\tau^2\ \sigma^2}\right]\right\}$$

which is a bivariate normal distribution. The conditional distribution $f(\Theta|y_n)$, therefore, is univariate normal with

mean

$$\frac{\frac{\mu\sigma^2}{n} + y_n\tau^2}{\tau^2 + \sigma^2/n}$$

and variance

$$\frac{\tau^2\sigma^2}{n\tau^2 + \sigma^2} .$$

Since the mean of Θ is assumed to be zero and $Y_n = 1/n \sum_{i=1}^{n} X_i$ the mean of $f(\Theta|y_n)$ reduces to

(7)

$$\frac{\tau^2(X_1^1+\ldots+X_n^1)}{\sigma^2 + \tau^2} .$$

The random sample $X_1^1,\ldots,X_n^1$ can be represented by a vector $X = (X_1^1,\ldots,X_n^1)'$ whose covariance matrix is,

$$V = \begin{pmatrix} \sigma^2 & 0 & \cdots & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & \sigma^2 \end{pmatrix} .$$

In order to find the joint density function $f_\theta(X_1^1,\ldots,X_n^1)$, it is sufficient to find its moment generating function.

$$E[e^{t'X}] = \int_{-\infty}^{\infty} e^{t'(X|\theta)}\, g(\theta)d\theta$$

$$= \int_{-\infty}^{\infty} \exp\left[t'\theta + \frac{t'Vt}{2}\right] \frac{1}{\sqrt{2\pi}\ \tau} \exp -\frac{1}{2}\left\{\frac{(\theta-\mu)^2}{\tau^2}\right\} d\theta$$

$$= \exp\left[\frac{t'Vt}{2}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\ \tau} \exp-\frac{1}{2}\left\{-2t'\theta + \frac{(\theta-\mu)^2}{\tau^2}\right\} d\theta .$$

Completing the square on the term inside the integral yields,

(8)

$$E\left[e^{t'X}\right] = \exp\left[\frac{t'Vt}{2} - \frac{\{\tau^2\sum_1^n t_i\}^2}{2\tau^2} + \mu\sum_1^n t_i\right]$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\tau} \exp -\frac{1}{2}\left[\frac{(\theta-(\tau^2\Sigma t_i+\mu))^2}{\tau^2}\right] d\theta$$

$$= \exp\left[\mu\sum_1^n t_i + \frac{t'Vt}{2} + \frac{\tau^2\sum_1 t_i^{\,2}+2\tau^2\sum_{i\neq j} t_i\cdot t_j}{2}\right]$$

$$= \exp\left[t'\mu + \frac{t'Vt}{2} + \frac{t'Zt'}{2}\right]$$

where

$$Z = \begin{bmatrix} \tau^2 & \cdot & \cdot & \cdot & \cdot & \tau^2 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \tau^2 & \cdot & \cdot & \cdot & \cdot & \tau^2 \end{bmatrix}$$

or

$$= \exp\left[t'\mu + \frac{t'Rt}{2}\right]$$

where

$$R = \begin{bmatrix} \sigma^2+\tau^2 & \tau^2 & \cdot & \cdot & \cdot & \cdot & \cdot\tau^2 \\ \tau^2 & \sigma^2+\tau^2\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\tau^2 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \tau^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\sigma^2+\tau^2 \end{bmatrix}.$$

Equation (8) is the moment generating function of a multivariate normal distribution with mean μ and covariance

matrix R. It should be noted that the random variables $X_1^1, \ldots, X_n^1$ are not jointly independent as was the case in model I.

$$\text{If } X = \begin{bmatrix} X_n^1 \\ \hline X_1^1 \\ \cdot \\ \cdot \\ \cdot \\ X_{n-1}^1 \end{bmatrix} = \begin{bmatrix} X_1 \\ \hline X_2 \end{bmatrix},$$

$$\mu = \begin{bmatrix} \mu \\ \hline \mu \\ \cdot \\ \cdot \\ \cdot \\ \mu \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hline \mu_2 \end{bmatrix},$$

and

$$R = \left[\begin{array}{c|c} \sigma^2+\tau^2 & \tau^2 \cdots\cdots \tau^2 \\ \hline \tau^2 & \sigma^2+\tau^2 \cdots\cdots \tau^2 \\ \cdot & \tau^2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \tau^2 & \tau^2 \cdots \sigma^2+\tau^2 \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right]$$

are partitioned as shown, the probability density function $f_\theta(X_n^1 | x_1^1, \ldots, x_{n-1}^1)$ is univariate normal, with

(9)
$$\text{mean } \mu_1 + \Sigma_{12}\,\Sigma_{22}^{-1}\,(X_2-\mu_2)$$

and

(10)
$$\text{variance } \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

As in model I, a decision rule that maximizes the conditional probability of hitting the target on each round is desired. From equation (10), the distribution of the n^{th}

round given the impact points of the previous (n-1) rounds is univariate normal. From Theorem 2, to maximize the conditional probability of hit on the n^{th} round, the mean of $f_\theta(X_n^n|x_1^{n-1},\ldots,x_{n-1}^{n-1})$ should be zero. Let ξ_n represent the shift that, when made prior to firing the (n+1)st round, results in the mean of $f_\theta(X_{n+1}^{n+1}|x_1^n,\ldots,x_n^n)$ being equal to zero. From equation (9) the mean of $f_\theta(X_{n+1}^{n+1}|x_1^n,\ldots,x_n^n)$ is

$$\begin{pmatrix} \tau^2,\ldots,\tau^2 \end{pmatrix}\begin{pmatrix} \sigma^2+\tau^2 & \tau^2\cdots\cdots,\tau^2 \\ \tau^2 & \sigma^2+\tau^2 \\ \cdot & \\ \cdot & \\ \cdot & \\ \tau^2\cdots\cdots\cdots & \sigma^2+\tau^2 \end{pmatrix}^{-1}\begin{pmatrix} x_1^n \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n^n \end{pmatrix} = T\ A\ X$$

where μ, the mean of the prior is zero.

The following development will show that the mean of $f_\theta(X_{n+1}^{n+1}|x_1^n,\ldots,x_n^n)$ is

$$\frac{\tau^2(x_1^n+\cdots+x_n^n)}{\sigma^2+n\tau^2}.$$

The matrix A can be put into the form,

$$A = \begin{pmatrix} D & -E\ \cdots\cdots\cdots & -E \\ -E & D\quad -E\ \cdots & -E \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ -E & \cdots\cdots\cdots\cdots & D \end{pmatrix}$$

where

$$D = \frac{\sigma^2+(n-1)\tau^2}{\sigma^4+n\sigma^2\tau^2}$$

and

$$E = \frac{\tau^2}{\sigma^4+n\sigma^2\tau^2}.$$

The proof will be by induction.

a. Show true for k=1. From equation (9) the mean is

$$(\tau^2)\ (1/\sigma^2+\tau^2)\ (x_1^1) = \tau^2 x_1^1/(\sigma^2+\tau^2).$$

b. Assume true for k=n-1.

c. Show true for k=n. From equation (9) the mean is $T'AX = c(x_1^n+\ldots+x_n^n)$ where,

$$\begin{aligned} c &= \tau^2(D-(n-1)E) \\ &= \tau^2 \left\{ \frac{\sigma^2+(n-1)\tau^2}{\sigma^4+n\sigma^2\tau^2} - \frac{(n-1)\tau^2}{\sigma^4+n\sigma^2\tau^2} \right\} \\ &= \frac{\tau^2}{\sigma^2+n\tau^2}\ . \end{aligned}$$

Therefore, the mean of $f_\theta(X_{n+1}^{n+1}|x_1^n,\ldots,x_n^n)$ is

(11)

$$T'AX = c(x_1^1+\cdots+x_n^n) = \frac{\tau^2(x_1^1+\ldots+x_n^n)}{\sigma^2 + n\tau^2} \quad \text{QED.}$$

In a similar manner equation (10) can be rewritten so that the variance of $f_\theta(X_{n+1}^{n+1}|x_1^n,\ldots,x_n^n)$ is,

(12)

$$\sigma^2+\tau^2 - \frac{n\tau^4}{\sigma^2+n\tau^2}\ .$$

It follows, therefore, that to maximize the conditional probability of hit on the (n+1)st round the decision-maker should make a shift $\xi_n = -\tau^2(x_1^n+\cdots+x_n^n)/(\sigma^2+n\tau^2)$ prior to firing the (n+1)st round. By Theorem 2 this action is optimal in terms of the criterion established for this mission.

The conditional probability of hit on the (n+1)st round is,

$$P(\text{hit}) = \int_{-a}^{a} f_\theta(X_{n+1}^{n+1} | x_1^n, \ldots, x_n^n)\, dx.$$

By equations (11) and (12) the conditional distribution $f_\theta(X_{n+1}^{n+1} | x_1^n, \ldots, x_n^n)$ is from the univariate normal family with a mean that is a function of the observed impact points $x_1^n, \ldots, x_n^n$, and whose variance varies with n but is independent of where the rounds actually impact. By assumptions three and four, Chapter II, the location of the rounds is known without error and when a shift is made it, too, is made without error. The knowledge of the exact impact points of the first n rounds allows a shift to be made that results in the mean of $f_\theta(X_{n+1}^{n+1} | x_1^n, \ldots, x_n^n)$ being equal to zero. This shift is made prior to firing the (n+1)st round. The conditional probability of hit on the (n+1)st round, therefore, is dependent on the previously fired rounds only through the change in variance resulting from the number of rounds previously fired. The conditional probability of hit, therefore, can be independent determined at each stage.

B. RESULTS

1. Decision Rule

From an analysis of model II, a decision rule requiring the shift $\xi_n = -\tau^2(x_1^n + \cdots + x_n^n)/(\sigma^2 + n\tau^2)$ at the n^{th} stage will maximize the conditional probability of hit. Additionally, by maximizing the conditional probability of hit for each round fired, the objective of minimizing the

expected number of rounds required to hit the target for the first time is achieved.

2. Independence

Since the conditional probability of hit for each round is independent of where the previous rounds actually impact, equation (4) of Theorem 1 applies to this model.

V. COMPARISON OF MODELS

A. MODEL I AND MODEL II

1. Bounds on E(J)

In this section an analysis of the two models will be made with a view toward determining which is the appropriate model to use in a given situation. Since the objective of the destruction mission is to destroy the target with the expenditure of the least number of rounds, the choice between models will be based on the expected number of rounds required to hit the target for the first time. Since $1-P_i$ represents the conditional probability that the target will not be hit on the i^{th} round given that it was not hit on the previous rounds, the probability that the number of rounds required to hit the target for the first time is greater than n is,

$$P(J>n) = \prod_{i=1}^{n} (1-P_i).$$

Additionally, since J is a non-negative integer valued random variable, the expected value of J, E(J), can be expressed as:

(13)

$$E(J) = \sum_{n=1}^{\infty} P(J>n).$$

It was shown in Chapters III and IV that P(J>n) can be independently determined regardless of where the previous rounds actually impact. Since, in general the individual

values of $1-P_i$, $i=1,\ldots,n$, are different for each i, a general explicit expression for the E(J) cannot be determined. It is possible, however, to establish bounds on E(J). Suppose P(J>n) is equal to a constant value R for all n. From equation (13), a lower bound of E(J) will occur when R is as small as possible while an upper bound will occur when R is at its maximum value. By proper choice of a constant R, upper and lower bounds for the E(J) can be determined for each model. This will be discussed in the next two sections.

2. Model I

From equation (6), and a shift $\xi_{n-1} = \frac{1}{n-1} x^{n-1}_{n-1}$,

(14)

$$1-P_n = \left\{ \Phi\left(\frac{a}{\sqrt{\sigma^2+\sigma^2/n-1}}\right) - \Phi\left(\frac{-a}{\sqrt{\sigma^2+\sigma^2/n-1}}\right) \right\}$$

$$= 2\left(1 - \Phi\left(\frac{a}{\sqrt{\sigma^2 \quad \sigma^2/n-1}}\right)\right)$$

where Φ is the standard normal distribution function and 2a is the size of the target.

Since in model I, $1-P_1=1$, equation (13) can be written in the form,

(15)

$$E(J) \simeq \sum_{n=0}^{\infty} R^n = \frac{1}{1-R} \; ; \; 0 < R < 1.$$

Since $P(J>n) = \prod_{i=1}^{n}(1-P_i)$, from Theorem 1, P(J>n) will be small when $(1-P_i)$ is small $\forall i$ and will be large when $(1-P_i)$ is large $\forall i$. From equation (14), $1-P_n$ will be as small as

possible when n is very large. For n large, $1-P_n \sim 2(1-\Phi(\frac{a}{\sigma}))$. If, therefore, $P(J>n)$ is set equal to $\prod_{i=1}^{n}(1-P_i)$ where $1-P_i = 2(1-\Phi(\frac{a}{\sigma})) = R$, equation (15) can be written as,

(16)

$$E_*(J) = \frac{1}{1-2(1-\Phi(\frac{a}{\sigma}))} = \frac{1}{2\Phi(\frac{a}{\sigma}) - 1}.$$

The value of $E_*(J)$ determined by equation (16) will be a lower bound for $E(J)$.

Similarly $1-P_n$ will be large when n=2 or $1-P_n = 2(1- (a/\sqrt{2\sigma^2}))$ and equation (15) can be written as,

(17)

$$E^*(J) = \frac{1}{1-2\left(1-\Phi\left(\frac{a}{\sqrt{2\sigma^2}}\right)\right)} = \frac{1}{2\Phi\left(\frac{a}{\sqrt{2\sigma^2}}\right) - 1}.$$

The value of $E^*(J)$ determined by equation (17) will be an upper bound for $E(J)$.

3. Model II

From equation (12), with a shift

$$\xi_{n-1} = \frac{-\tau^2(x_1^{n-1}+\ldots+x_{n-1}^{n-1})}{\sigma^2+(n-1)\tau^2},$$

(18)

$$1-P_n = 2\left(1-\Phi\left(\frac{a}{\sqrt{\sigma^2 \quad \tau^2 - \frac{(n-1)\tau^4}{\sigma^2+(n-1)\tau^2}}}\right)\right).$$

Equation (13) can be written in the form,

(19)

$$E(J) \simeq \sum_{n=1}^{\infty} R^n = \frac{R}{1-R}, \quad 0 < R < 1.$$

From equation (18), $1-P_n$ will be small when $\tau=0$ and large when $n=1$. As for model I upper and lower bounds for $E(J)$ can be established. For the lower bound, setting $R = 1-P_n = 2(1-\Phi(\frac{a}{\sigma}))$, equation (19) can be written as

(20)

$$E_*(J) = \frac{2(1-\Phi(\frac{a}{\sigma}))}{2\Phi(\frac{a}{\sigma})-1} .$$

Similarly for the upper limit, setting $R = 1-P_n = 2(1-\Phi(a/\sqrt{\sigma^2+\tau^2}))$, equation (19) can be written as,

(21)

$$E^*(J) = \frac{2(1-\Phi(\frac{a}{\sqrt{\sigma^2+\tau^2}}))}{2\Phi(\frac{a}{\sqrt{\sigma^2+\tau^2}})-1} .$$

4. Comparison

An analysis of the upper and lower bounds for $E(J)$ revealed the following. For the lower limit, the values given by equations (16) and (20) do not depend on τ^2, the variance of the prior distribution of Θ. The denominators of these equations are the same whereas the numerator of model I is equal to one and that of model II is less than one. This implies that the lower bound for model II is less than the lower bound for model I. For the upper bound, however, a comparison of equations (17) and (21) is dependent on τ^2. When $\tau^2=0$, $E^*(J)$ for model II is less than that for model I, whereas when τ^2 is large, $E^*(J)$ for model I is less than that for model II.

To illustrate, consider the following example. Let $a=10$ meters and $\sigma=12$ meters. The bounds of $E(J)$ as a function

of τ^2 are shown in Figure 11. When $\tau^2=280$, $E^*(J)$ for model II is equal to $E_*(J)$ for model I so that for values of τ^2 under region I of Figure 11, model II results in the smallest $E(J)$. When $\tau^2=496$ $E^*(J)$ for model II is equal to $E^*(J)$ for model I so that for values of τ^2 under region III model I results in the smallest $E(J)$. For values of τ^2 in region II a degree of uncertainty exists in regard to which model results in the smallest $E(J)$.

As stated in Chapter II, the expected number of rounds required to hit the target for the first time is the criterion for choice among models. From a strictly mathematical point of view, therefore, model I will be preferred when τ^2 falls in region I, whereas model II will be preferred when τ^2 falls in region III of Figure 11. For values of τ^2 in region II, the choice must be based on the decision-makers past experience with the models and his subjective analysis of the situation.

Beyond the mathematics of the problem are the following questions of interpretation. If the decision-maker truly believes in the Bayes approach of model II is he ever justified in using model I? Conversely, if the decision-maker does not believe that the value of θ is a random variable, is he ever justified in using model II? Questions of this nature are at the heart of the controversy between the Bayesian and classical approach to statistical decision making. The author chooses not to blindly adhere to either the Bayes or classical school of thought but rather

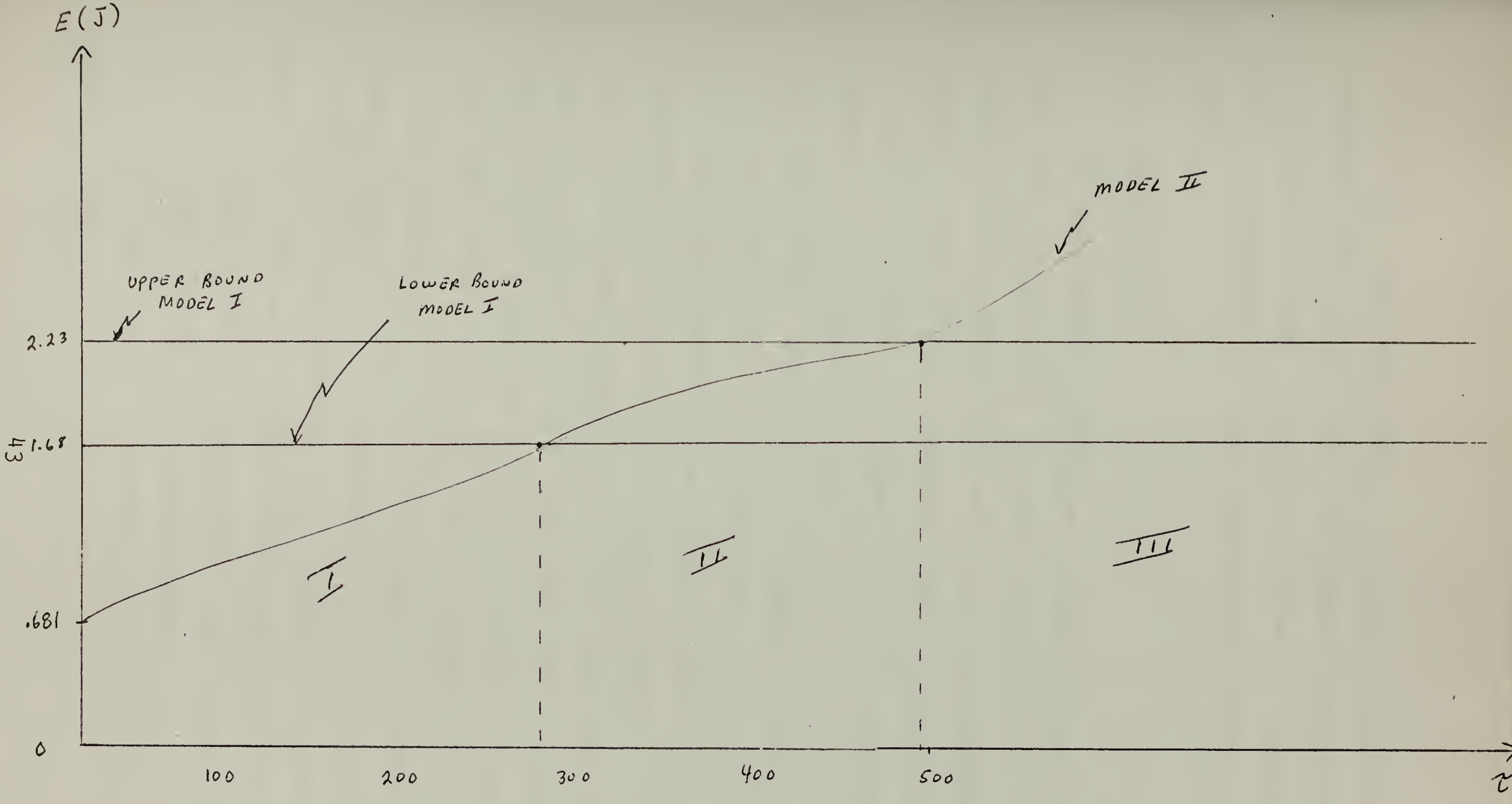
E(J)
UPPER BOUND
MODEL I
LOWER BOUND
MODEL I
MODEL II
2.23
1.68
.681
0
100
200
300
400
500
I
II
III
τ

FIGURE 11

to use the model that, under a particular circumstance, results in the smallest E(J). The choice between models, therefore, is based on the E(J) produced by the model and not on the school of thought that is the foundation of the model.

B. GRUBBS MODEL

The problem of adjusting on a target by artillery fire has been addressed by F. E. Grubbs, [Reference 6]. In his report, Grubbs developed two models for the destruction mission. His criterion, however, was to minimize the variance of the center of impact of the n^{th} round fired. His first model required a shift of $\xi_n = \frac{-1}{n} x_n^n$ to be made after each round. This shift corresponds to that of model I of this paper. This shift, therefore, in addition to minimizing the variance of the center of impact of the n^{th} round also maximizes the conditional probability of hit on the n^{th} round and minimizes the expected number of rounds required to hit the target for the first time. His second model required a shift of $\xi_n = -\tau^2(x_n^n)/(\sigma^2+n\tau^2)$ to be made after each round. This shift does not maximize the conditional probability of hitting the target or minimize the expected number of rounds required to hit the target for the first time (see equation (11)). The choice between model II of this paper and the second model developed by Grubbs depends on whether the decision-maker wants to minimize the expected number of rounds required to hit the target for the first time or to minimize the variance of the center of impact of the n^{th} round.

VI. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

If the objective of the destruction mission is to minimize the expected number of rounds required to hit a target for the first time, a procedure that will maximize the conditional probability of hit, for each round fired, should be used. Analysis of the models developed in this paper results in decision rules consistent with this objective.

As indicated in Chapter I, the procedure currently in use by the Army does not make use of observer estimates of how far each round impacts from the target nor does it use information concerning the impact points of all previously fired rounds. The introduction of the laser range-finder should allow a more accurate observer estimate of miss distance than could be provided without this piece of equipment. Additionally, TACFIRE, with its ability to store data and make high speed accurate calculations, admits the possibility of computing firing data based on all available information without significant increase in mission time. Assuming the capabilities of the laser range-finder and TACFIRE, models I and II allow the decision-maker to base his decision on more information than is the case in the current procedure.

For each size target, gun-target range, and prior distribution on Θ, the best model to use is determined by E(J)

for each model. The model with the smallest E(J) is preferred. As shown in Figure 11, however, there is an overlap region where the decision-maker must rely on his past experience in order to choose between the two models.

B. AREAS FOR FURTHER RESEARCH

Both models utilize the assumption that there is no observer or crew error. Through computer simulation the sensitivity of these two models to such errors could be assessed. Additionally, a comparison between the two models and the current procedure could be valuable.

The choice between the two models depends to a great extent on the value of τ^2, the variance of the prior distribution of Θ. Experimentation should be conducted to determine values of τ^2 for various ranges, firing conditions, and weapon systems. Additionally, the realism of the assumption of a prior distribution from the normal family for model II should be verified.

C. RECOMMENDATIONS

It is recommended that the Army continue to investigate alternative procedures for the conduct of the destruction mission. Technologocal improvements that are currently being introduced to the field artillery suggest a consideration of procedures that provide more information that is currently provided in the decision making situation. The models presented in this paper are possible alternatives.

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13. ABSTRACT

The purpose of this thesis is to mathematically model the Field Artillery Destruction Mission. The author felt that advances in technology might allow the development of procedures that are more efficient than those currently in use. In particular TACFIRE, a computer based fire direction center, and the laser range-finder were taken into consideration. Using the capabilities resulting from these technological advances, a classical and Bayesian model of the destruction mission was developed. Each model was analyzed and conclusions were drawn regarding the appropriate model to use in a given situation.

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